Effective dynamics of a classical point charge

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The effective Lagrangian of a point charge is derived by eliminating the electromagnetic field within the framework of the classical closed time path formalism. The short distance singularity of the electromagnetic field is regulated by an UV cutoff. The Abraham-Lorentz force is recovered and its similarity to anomalies is underlined. The full cutoff-dependent linearized equation of motion is obtained, no runaway trajectories are found but the effective dynamics shows acausality if the cutoff is beyond the classical charge radius. The strength of the radiation reaction force displays a pole in its cutoff-dependence in a manner reminiscent of the Landau-pole of perturbative QED. Similarity between the dynamical breakdown of the time reversal invariance and dynamical symmetry breaking is pointed out.

I. INTRODUCTION

Classical electrodynamics of point charges has an intrinsic length scale, the classical charge radius $r_0 = e^2/mc^2$, separating the well known classical domain from an unusual classical world, hidden behind quantum effects. Nevertheless it is an intriguing question how this classical world of point charges would look like in the absence of quantum mechanics. The attempts to uncover the dynamics around of or beyond to the crossover scale, r_0, are rendered difficult by the current state of understanding of the radiation reaction force [1], an important ingredient of the electromagnetic interaction at these scales. The problem comes from the singularity of the electromagnetic field (EMF) in the point-like limit of the charges and appears as an the instability, generated by the Abraham-Lorentz force, the runaway solution [2].

A simple, systematic derivation of the radiation reaction is presented here by working out the effective dynamics of a single point charge. The singularities of the point charge limit are regulated by smearing the interactions within an invariant length ℓ, an UV cutoff, and the dissipative nature of the radiation reaction force is handled by an extension of the variational method of classical mechanics, by recasting Schwinger's the Closed Time Path (CTP) formalism [3]. Thought this scheme has already been applied in a number of problems its advantages and power has not yet been exhausted. It has been used to find relaxation in many-body quantum systems [5], to generate perturbation expansion for retarded Green functions in quantum field theory [6], to find manifestly time reversal invariant description of quantum mechanics [7], to describe finite temperature effects in quantum field theory [8], to find the mixed state contributions to the density matrix by path integral [9], to describe non-equilibrium processes [10], to derive equations of motion for the expectation value of local operators [11], and to describe scattering processes with non-equilibrium final states [12]. This scheme, used here in field theory [4], provides a compromise which on the one hand, preserves the functional formalism of the variational principle, a natural way to introduce Green functions and perturbation expansion, and on the other hand, it allows the imposition of initial conditions and dissipative forces.

The well known expression of the Abraham-Lorentz force is recovered in the calculation, together with the usual, UV divergent mass renormalization in the limit $\ell \to 0$. The linearized nonlocal equation of motion is derived for a point charge by retaining the full cutoff dependence. It is pointed out that we do not possess all data necessary to solve an initial condition problem and the solution is constructed in terms of the retarded Green function. The mode which drives the usual runaway solution is absent and the motion is causal for $\ell \gg r_0$. For sufficiently small values of ℓ unstable modes appear but they do not lead to runaway solutions because the retarded Green function relies on the unstable modes in the acausal regime where these modes are bounded.

The organization of this paper is the following. Section II introduces the CTP formalism in classical field theory which gives a systematic definition of the retarded Green function in classical effective theories. The way the time arrow is generated by the environment, irreversibility, acausality arise and the runaway solution is avoided are the subject of Section III. Section IV contains the derivation of the effective equation of motion. Finally, the summary of the results is presented in Section V. Some details about the CTP Green function are collected in Appendix A.

II. CTP

An extension of the variational method of classical mechanics is needed to cover dissipative forces in a functional framework where the solution of initial condition problems can be found by means of a systematically derived retarded Green function. We consider a classical system described by the coordinate x, governed by the Lagrangian $L(x, \dot{x})$.

A solution of the equation of motion for $t_i \le t \le t_f$ is usually identified by imposing auxiliary conditions, such as $x(t_i) = x_i$ and $x(t_f) = x_f$. Can we replace these boundary conditions with the initial conditions $x(t_i) = x_i$, $\dot{x}(t_i) = v_i$? The problem is that the equation of motion must then be imposed at the final time and it cancels the generalized momentum. This condition can be avoided by constructing a particular trajectory, $\tilde{x}(\tilde{t})$, for twice as long time interval, $0 < \tilde{t} < 2(t_f - t_i)$ which satisfies the desired initial conditions. The particularity of the trajectory is that a time reversal transformation is performed at $\tilde{t} = t_f - t_i$, first the change $\dot{\tilde{x}}(t_f - t_i) \to -\dot{\tilde{x}}(t_f - t_i)$ is performed and then the same equation of motion is solved backward in time. The motion stops at $\tilde{t} = 2(t_f - t_i)$ when $\tilde{x}$ arrives at the time reversed initial conditions. A more convenient book-keeping which is used below consists of a formal reduplication of the degrees of freedom, $\tilde{x} \to \hat{x} = (x^+, x^-)$ where the members of the CTP doublet are defined for $t_i \le t \le t_f$ as $x^+(t) = \tilde{x}(t - t_i)$ and $x^-(t) = \tilde{x}(2t_f - t_i - t)$.

It will be important to distinguish true time reversal transformation from a formal reparametrization of the motion because the auxiliary conditions are handled differently. Both involve the reversal of the direction of time in the equation of motion and the exchange of the initial and final time, $t_i \leftrightarrow t_f$, in the auxiliary conditions. The values of x_i and v_i are transformed as $x_i \to x_f$ and $v_i \to -v_f$ in time reversal and the solution changes if the equation of motion is not time reversal invariant. In the time reversed reparametrization the values of x_i and v_i are adjusted to recover the same trajectory in reversed time, irrespectively of the time reversal properties of the equation of motion.

A. Finite time motion

The variational principle is based on the action

$$S_{CTP}[\hat{x}] = \int_{t_i}^{t_f} dt[L_{CTP}(x^+(t), \dot{x}^+(t)) - L^*_{CTP}(x^-(t), \dot{x}^-(t))], \tag{1}$$

where $\mathrm{Re}L_{CTP} = L$ is the original Lagrangian. The variation is within the set of trajectories defined by the auxiliary conditions, namely both members of the CTP doublet satisfy the same intial conditions, $x^\pm(t_i) = x_i$, $\dot{x}^\pm(t_i) = v_i$, and the constraint

$$x^+(t_f) = x^-(t_f). \tag{2}$$

This latter assures that the boundary term, arising in the calculation of the equation of motion from the variation at the final time cancels. The opposite signs in front of the two Lagrangians is to render the variational equation trivial, $0 = 0$, at the final time as far as the real part of the action is concerned. If the original Lagrangian were used for both trajectories then the CTP action would be degenerate for the CTP doublets $x^+(t) = x^-(t)$. This degeneracy can be lifted by introducing an infinitesimal difference between the forward and backward running dynamics in time. The simplest difference, introduced between the two time axis which switches off adiabatically the auxiliary conditions in the limit $t_f - t_i \to \infty$ as expected corresponds to

$$L_{CTP}(x, \dot{x}) = L(x, \dot{x}) + \frac{i\epsilon}{2}x^2 \tag{3}$$

and the limit $\epsilon \to 0$ is supposed to be performed after deriving and solving the variational equations of motion as in Feynman $i\epsilon$ prescription in the quantum case. Note that the formal reparametrization of the motion, called CTP transformation below, generates $S_{CTP} \to -S^*_{CTP}$ and preserves the equation of motion.

B. Green function

To find the Green function we consider a harmonic system, defined by the action

$$S_{CTP}[\hat{x}] = \int dt \left[\frac{1}{2}\hat{x}(t)\hat{K}\hat{x}(t) + \hat{j}(t)\hat{x}(t)\right], \tag{4}$$

and the CTP Green function, $\hat{D} = \hat{K}^{-1}$ yields the trajectory

$$\hat{x}(t) = -\int dt' \hat{D}(t, t')\hat{j}(t'), \tag{5}$$

written as $\hat{x} = -\hat{D}\hat{j}$ in condensed notation. The $\mathcal{O}(\epsilon)$ term in the CTP action makes the null-space of $\hat{K}$ trivial and its inverse, $\hat{D}$, well defined.

The CTP transformation, $S_{CTP} \to -S^*_{CTP}$, under time reversed reparametrization requires the block structure

$$\hat{K} = \sigma \begin{pmatrix} K^n + iK^i_1 & -K^f + iK^i_2 \\ K^f + iK^i_2 & -K^n + iK^i_1 \end{pmatrix} \sigma, \tag{6}$$

where K^i_1, K^i_2, K^n and K^f are real functions and

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7}$$

denotes the CTP "metric tensor".It is easy to see that the inverse has a similar block structure,

$$\hat{D} = \begin{pmatrix} D^n + iD^i_1 & -D^f + iD^i_2 \\ D^f + iD^i_2 & -D^n + iD^i_1 \end{pmatrix}. \tag{8}$$

including four real functions D^i_1, D^i_2, D^n and D^f. The quadratic form (6) and its inverse (8) are symmetric, $\hat{D}^{tr} = \hat{D}$, therefore the relations $(D^i_1)^{tr} = D^i_1$, $(\tilde{D}^i_2)^{tr} = D^i_2$, $(\tilde{D}^n)^{tr} = D^n$, $(D^f)^{tr} = -D^f$ follow.

Though we introduced two independent sources, $\hat{j} = (j^+, j^-)$ to diagnose the forward and backward moving part of the dynamics independently, the physical external source corresponds to the particular choice $\hat{j} = (j, -j)$. The solution of the equation of motion can be written as

$$x^\sigma(t) = -\sum_{\sigma'=\pm1} \int_{t_i}^{t_f} dt' D^{\sigma\sigma'}(t,t')\sigma' j(t') \tag{9}$$

in this case and to find a σ-independent solution we need the identity

$$D^{++} + D^{--} = D^{-+} + D^{+-} \tag{10}$$

which imposes $D^i_1 = D^i_2$,

$$\hat{D} = \begin{pmatrix} D^n + iD^i & -D^f + iD^i \\ D^f + iD^i & -D^n + iD^i \end{pmatrix}. \tag{11}$$

The retarded and advanced Green functions are defined as $D^{\overset{r}{a}} = D^n \pm D^f$. Causality, the effect of an external perturbation appearing after the time of the action of the perturbation further reduces the number of independent functions and requires $D^f(t,t') = \epsilon(t-t')D^n(t,t')$. It is advantageous to define the retarded and advanced combinations of the quadratic form of the action, $K^{\overset{r}{a}} = K^n \pm K^f$ and elementary algebra gives the relations [13]

$$K^{\overset{r}{a}} = D^{\overset{r}{a}-1}, \tag{12}$$

and $K^i_1 = K^i_2 = K^i$ with

$$K^i = -D^{r-1} D^i D^{a-1}. \tag{13}$$

The choice of the imaginary part of the action remains free in classical physics for this part of the Green function controls quantum fluctuations only. The explicit calculation of the Green function is outlined in Appendix A. Time reversal flips the direction of the time in which the external source is visited during the motion and it is represented by the transformation $\hat{D}(x,x') \to \hat{D}(Tx,Tx')$, where $T(t,\boldsymbol{x}) = (-t,\boldsymbol{x})$ and induces $D^f \to -D^f$, $D^r \leftrightarrow D^a$ and $K^f \to -K^f$.

C. Infinite time motion

The action (1) and the auxiliary conditions describes the dynamics in a finite time interval and its Green function is not invariant under translations in time. The extension of the action principle is not obvious for $t_f - t_i \to \infty$ because the auxiliary condition (2) must be preserved or else the two members of the CTP doublets decouple. But the action can be constructed in this limit by first calculating the Green function for $t_f - t_i \to \infty$ and defining the quadratic form of the action as the inverse of the Green function. Let us consider a free scalar field, described by the action

$$S_{CTP}[\hat{\phi}] = \int dx \left[\frac{1}{2}\hat{\phi}(x)\hat{K}\hat{\phi}(x) + \hat{j}(x)\hat{\phi}(x) \right]. \tag{14}$$

The Green function is calculated in Appendix A and its inverse, given by Eq. (A24) leads to

$$S_{CTP}[\hat{\phi}]=\frac{1}{2}\int\frac{dp}{(2\pi)^4}(\phi^+(-p),\phi^-(-p))\begin{pmatrix}p^2-m^2+i\epsilon & -2i\epsilon\Theta(-p^0)\\ -2i\epsilon\Theta(p^0) & -p^2+m^2+i\epsilon\end{pmatrix}\begin{pmatrix}\phi^+(p)\\ \phi^-(p)\end{pmatrix}, \tag{15}$$

and $K^n=p^2-m^2$, $K^f=i\mathrm{sign}(p^0)\epsilon$ and $K^i=\epsilon$.

Few remarks are in order at this point. (i) Translation invariance in time is recovered in (15) by the adiabatic switching off the auxiliary conditions by the imaginary part (3) of the action. (ii) Note the transmutation of the coupling between the two time axes: The identification of the final coordinates of the two members of the CTP doublet for the finite t_f-t_i is traded into a coupling of infinitesimal, $\mathcal{O}(\epsilon)$ strength for $-\infty<t<\infty$ in the action (15). (iii) The trajectory, given by the retarded Green function in the limit $t_f-t_i\to\infty$ corresponds to the trivial, $x_i=v_i=0$ initial conditions, imposed in the distant past.

D. Effective theory

The effective system dynamics is generated by the elimination of the environment variables by means of their equations of motion and auxiliary conditions. This procedure is now followed in a functional formalism which allows us to identify the retarded Green function in a systematic manner, without adjusting the pole structure by hand.

Let us introduce a system and environment coordinates, x and y, respectively and write the original multi-component coordinate as $X=X(x,y)$. The effective system dynamics is found by following the usual procedure of quantum field theory: One introduces a book-keeping source, j, and constructs the Legendre transform

$$W[j]=S[x,y]+\int dtj(t)x(t), \tag{16}$$

where $S[x,y]=S[X(x,y)]$ and y satisfies the equation of motion $\delta S[x,y]/\delta y=0$, equipped by some auxiliary conditions which make the solution unique. As a result, the equation $\delta W[j]/\delta j=x$ follows and the effective action is defined by the inverse Legendre transform,

$$S_{eff}[x]=W[j]-\int dtj(t)x(t). \tag{17}$$

The relation

$$\frac{\delta S_{eff}[x]}{\delta x}=-j, \tag{18}$$

suggests the interpretation of $S_{eff}[x]$ as effective system action. The construction makes it evident that this elimination is achieved by solving the equation of motion for the environment coordinates in (16) and recalculating the action in (17) in terms of the system coordinates. The advantage of the functional definition of the effective action is the simple definition of the effective Green function,

$$D=\left(\frac{\delta^2 S_{eff}[0]}{\delta x\delta x}\right)^{-1}. \tag{19}$$

The same procedure can be repeated within the CTP formalism where the CTP symmetry allows us to write the effective action as

$$S_{eff}[\hat{x}]=S_0[x^+]-S_0^*[x^-]+S_I[\hat{x}]. \tag{20}$$

The definition of the influence functional S_I [9], the coupling the two time axes is rendered unique by imposing $\delta^2 S_I[x^+,x^-]/\delta x^+\delta x^-\neq 0$.

It is illuminative to separate the system-environment interactions into two classes. An interaction is called closed if its energy-momentum distribution is localized in space-time around the system, for instance the Coulomb field of a charge or a polaron in solids. Such a localized interactions keep the system closed and can be represented as a renormalization of the system properties and contribute to S_0. The open interactions create environment excitations which leave the vicinity of the system. They render the system dynamics open, make the system energy-momentum non-conserved and give contributions to S_I. The distinguishing feature of the CTP formalism is this mapping of system-environment interactions into interactions in between two copies of the same physical system. This mapping generates system-environment entanglement in the quantum case.

III. TIME ARROW

We address here the breakdown of time reversal symmetry and the causal structure furthermore the fate of unbounded, runaway trajectories in effective theories. The effective dynamics where the time reversal invariance is broken dynamically by the environment is irreversible because dynamical symmetry breaking implies infinitely many weakly coupled environment degrees of freedom and allows the application of the methods of statistical physics. Causality is a logical relation, namely A is the cause of B if the appearance of A always leads to B. The runaway modes will be sought in harmonic systems where they grow exponentially in time. All these are related to the time arrow.

A possible experimental determination of the time arrow is the application of an external perturbation which is localized in time: The time arrow is forward or backward if the response is found after or before the perturbation, respectively. The time arrow, established in this manner is relative, it is defined with respect to the flow of the proper time of the observer. The time arrow is set in our calculations of the trajectory of a local equation of motion by imposing initial or final auxiliary conditions. One may go in thought beyond experimental limitations and diagnose the calculated trajectory either by imposing opposite time arrows on different degrees of freedom or seek the dynamical generation of time arrows, independent of auxiliary conditions.

A. Time reversal in effective dynamics

The distinguishing feature of time inversion, compared to other space-time symmetry transformation is the important role of auxiliary conditions in breaking time reversal invariance. Though this is an obvious issue for closed, autonomous dynamics, the fate of time reversal invariance is nontrivial in the effective dynamics of open systems where the auxiliary conditions of the undetected environment are build into the effective dynamics in an implicit manner.

In fact, let us denote the system and environment coordinates by x and y, respectively, and suppose that they obey time reversal and time translation invariant equations of motion,

$$\ddot{x} = F(x, y), \quad \ddot{y} = G(x, y), \tag{21}$$

with separable initial conditions, $f(x(t_i), \dot{x}(t_i)) = g(y(t_i), \dot{y}(t_i)) = 0$. The effective system dynamics is then constructed by first solving the environment equation of motion for an arbitrary environment trajectory, $y = y[x, g]$, and then inserting this solution into the system equation of motion,

$$\ddot{x} = F(x, y[x, g]). \tag{22}$$

Time reversal and time translations are realized in a nontrivial manner by the effective equation of motion due to the presence of the environment auxiliary condition $g(y)$. Though trivial auxiliary conditions can be suppressed adiabatically for free system as in Eq, (15), interactions usually make the nontrivial auxiliary conditions important, eg. integrable dynamics.

There is a simplification, acting for large enough environment, namely the relaxation to an equilibrium state. We perform in this case first the thermodynamical limit, $N \to \infty$ where N denote the number of weakly coupled environmental degrees of freedom and after that the limit $t_f - t_i \to \infty$. Though the latter decouples the environment auxiliary conditions and restores time translation invariance nevertheless time reversal invariance may remain broken. An effective dynamics is called irreversible if its effective dynamics recovers time translation invariance but remains noninvariant under time reversal. Irreversible dynamics with relaxation displays runaway, self-accelerating solutions after (system) time inversion.

We have no empirical basis for absolute time arrow or causality in the absence of a "control Universe" to compare phenomena in the presence or absence of an external perturbation. But relaxation to an equilibrium and irreversibility represent a practical approximation of the complicated system dynamics where the time arrow becomes absolute. It is based on the double limit limit $N \to \infty$ and $t_f - t_i \to \infty$ which allows a stable time evolution in one direction of the time only and remains applicable up to time scales far beyond reasonable observation length. Though this approximation scheme yields a unique time arrow for infinite systems causality remains elusive for short time processes. This will be demonstrated within a simple harmonic toy model.

B. Toy model

A simple but instructive toy model can be made for the system and environment coordinates, x and y_n, $n = 1, \ldots, N$, respectively, by introducing the Lagrangian [14]

$$L = \frac{m}{2}\dot{x}^2 - \frac{m\omega_0^2}{2}x^2 + jx + \sum_n \left(\frac{m}{2}\dot{y}_n^2 - \frac{m\omega_n^2}{2}y_n^2 - g_n y_n x\right), \tag{23}$$

where $\omega_n \geq 0$ and the bound $\sum_n g_n^2/\omega_n^2 < m^2\omega_0^2$ is required to have bounded potential energy and real frequency spectrum. We use the time arrow $\tau = 1$ or $\tau = -1$ to indicate that the trivial auxiliary conditions, ie. vanishing coordinate and velocity are specified at $t = t_i$ or $t = t_f$, respectively. All environment degree of freedom have the same time arrow. The effective system equation of motion,

$$-j(\omega) = [m(\omega^2 - \omega_0^2) - \Sigma_{\tau_e}(\omega)]x(\omega), \tag{24}$$

contains the self energy,

$$\Sigma_{\tau_e}(\omega) = \sum_n \frac{g_n^2}{m}\frac{1}{(\omega + i\tau_e\epsilon)^2 - \omega_n^2}, \tag{25}$$

and the retarded system Green function is given by

$$D_{\tau_s,\tau_e}(t) = \int \frac{d\omega}{2\pi}\frac{e^{-i\omega t}}{m[(\omega + i\tau_s\epsilon)^2 - \omega_0^2] - \Sigma_{\tau_e}(\omega)}. \tag{26}$$

It is easy to see that the poles of the Green function of discrete spectrum have infinitesimal negative imaginary part for consistent time arrows, $\tau_s = \tau_e$, making the effective dynamics reversible and causal. Conflicting time arrows, $\tau_s = -\tau_e$, may induce acausality.

In case of a spectrum with condensation point it is more advantageous to use the spectral function

$$\rho(\Omega) = \sum_n \frac{g_n^2}{2m\omega_n}\delta(\omega_n - \Omega), \tag{27}$$

and the self energy

$$\Sigma_{\tau_e}(\omega) = \int d\Omega \frac{2\rho(\Omega)\Omega}{(\omega + i\tau_e\epsilon)^2 - \Omega^2}. \tag{28}$$

Spectra with condensation point induces nonlocal equation of motion and make the issues of reversibility, causality and runaway modes nontrivial. The integration is passing a pole at infinitesimal distance on the complex Ω-plane and odd powers of $i\omega$ can be generated in the self energy with sign proportional to environment time arrow, $\mathrm{sign}(\tau_e)$. For instance a simple Ohmic spectral function with smooth suppression at high frequency,

$$\rho(\Omega) = \frac{\Theta(\Omega)g^2\Omega}{m\Omega_D(\Omega_D^2 + \Omega^2)}, \tag{29}$$

yields

$$\Sigma_{\tau_e}(\omega) = -\frac{g^2\pi}{m\Omega_D}\frac{\Omega_D + i\mathrm{sign}(\tau_e)\omega}{\omega^2 + \Omega_D^2}, \tag{30}$$

and an $\mathcal{O}\left(\Omega_D^{-1}\right)$ Newtonian friction force is found in the effective equation of motion (24) in for large cutoff, $\Omega_D \gg \omega_0$. The appearance of $\mathrm{sign}(\tau_e)$ in $\mathrm{Im}\Sigma_{\tau_e}(\omega)$ indicates irreversibility and the transmutation of the environment time arrow to the system. Furthermore there is no protection against having poles of the retarded Green function on the "wrong" sheet even for $\tau_s = \tau_e$ and the effective dynamics might be acausal [15]. When this happens then the real part of the pole represents the frequency scale at which acausality manifests itself. This scale is $\mathcal{O}(\Omega_D^{2/3})$ for the Ohmic spectral function and acausality appears at shorter time scale than the characteristic time of the oscillator or the time scale of the friction. It is easy to check that the retarded Green function remains bounded despite the "wrong" sign of the imaginary part of the pole.

C. Effective dynamics in CTP scheme

The retarded Green function of continuous spectrum displays several nontrivial features and calls for a more systematic construction. This can be achieved within the CTP formalism where the action of our toy model is

$$S[\hat{x},\hat{y}] = \frac{1}{2}\hat{x}\hat{D}_0^{-1}\hat{x} + \frac{1}{2}\sum_n \hat{y}_n\hat{G}_n^{-1}\hat{y}_n + \hat{x}\left(\hat{j} - \sigma\sum_n g\hat{y}_n\right), \tag{31}$$

and the inverse Green functions $\hat{D}_0^{-1}$ and $\hat{G}^{-1}$ are given by Eq. (A18), containing the appropriate frequency. The effective action

$$S_{eff}[\hat{x}] = \frac{1}{2}\hat{x}\hat{D}^{-1}\hat{x} + \hat{j}\hat{x}, \tag{32}$$

is given in terms of the Green function $\hat{D} = [\hat{D}_0^{-1} - \sigma\hat{\Sigma}\sigma]^{-1}$ where the self energy, $\hat{\Sigma} = \sum_n g_n^2\hat{G}_n$, has the structure of the right hand side of Eq. (8), namely

$$\begin{aligned}
\Sigma^n(\omega) &= \frac{1}{m}\sum_n g_n^2\frac{\omega^2-\omega_n^2}{(\omega^2-\omega_n^2)^2+\epsilon^2},\\
\Sigma^f(\omega) &= -\frac{i\epsilon\tau_e\mathrm{sign}(\omega)}{m}\sum_n\frac{g_n^2}{(\omega^2-\omega_n^2)^2+\epsilon^2},\\
\Sigma^i(\omega) &= -\frac{\epsilon}{m}\sum_n\frac{g_n^2}{(\omega^2-\omega_n^2)^2+\epsilon^2}.
\end{aligned} \tag{33}$$

The effective equation of motion will be sought in the parametrization $x^{\pm} = x \pm x^d/2$ of the CTP trajectories and write the effective action (32) as

$$S[\hat{x}] = x^d(D^{r-1}x + j), \tag{34}$$

where

$$D^r = \frac{1}{D_0^{-1} - \Sigma^n - \Sigma^f}, \tag{35}$$

with $D_0^{-1} = -\partial_t^2 - \omega_0^2$. D^r is the retarded system Green function according to Eq. (12). The equation of motion for x^d and x yields $x = -D^r j$ and $x^d = 0$, respectively.

D. Dynamical breakdown of time reversal invariance

Irreversibility, the loss of partial time reversal invariance by the environment auxiliary conditions is a dynamical symmetry breaking, driven by the impact of the environmental auxiliary conditions on the system dynamics. It is instructive to compare this state of affairs with the spontaneous symmetry breaking of a Z_2 symmetry which can be detected in two different manners. One possibility is dynamical, the following of the time dependence of an order parameter and observing its slowing down and becoming frozen at a finite value in the thermodynamical limit. Another, simpler possibility is to inspect the static order parameter in the presence of an explicit symmetry breaking and verify that it assumes a finite proportional to the sign of the explicit symmetry breaking term even when this latter is infinitesimal. Both signatures of spontaneous breakdown of partial time reversal symmetry can be found in irreversible systems.

The symmetry of the effective equation of motion with respect to time reversal is broken explicitly by the environment initial conditions. These initial conditions are first converted within the environment into an infinitesimally weak dynamical breakdown of symmetry as $t_f - t_i \to \infty$, acting during the time evolution, as pointed out after Eq. (15). In the second step, when the effective dynamics is constructed then this explicit, infinitesimal symmetry breaking by $K^f = \mathcal{O}(\epsilon)$ in the action (14) gives rise to a finite self energy term Σ^f in the effective action (32) when the spectral representation

$$\hat{\Sigma}_{\tau_e}(\omega) = \int d\Omega 2\Omega\rho(\Omega)\hat{D}(\tau_e\omega,\Omega) \tag{36}$$

is used with

$$\hat{D}(\omega,\Omega)=\frac{1}{m}\begin{pmatrix}\frac{1}{\omega^2-\Omega^2+i\epsilon} & -2\pi i\Theta(-\omega)\delta(\omega^2-\Omega^2)\\ -2\pi i\Theta(\omega)\delta(\omega^2-\Omega^2) & -\frac{1}{\omega^2-\Omega^2-i\epsilon}\end{pmatrix}. \tag{37}$$

cf. (A17). The signs of K^f and Σ^f are correlated, $\mathrm{sign}(K^f(\omega))\mathrm{sign}(\Sigma^f(\omega))=-1$, according to Eq. (12).

The thermodynamical limit, carried out in the dynamical test of spontaneous symmetry breaking is the limit of infinitely many environment degrees of freedom, $N\to\infty$, in our model. The slowing down of an order parameter can be recognized by the non-commutativity of the thermodynamical and the long observational time limits. The toy model with condensation point in its spectrum displays such a non-commutativity. On the one hand, arbitrarily large but finite system appears discrete for $t_f-t_i=\infty$. On the other hand, there is no observation within a finite amount of time which could resolve the spectrum with $N=\infty$ around a condensation point [15] and we have to rely on the continuous spectrum formalism. The infinitely many normal modes belonging to the unresolved part of the spectrum represent a sink for the system energy and may generate dissipative forces.

E. Causality

The equations of motion only establish correlation between dynamical quantities at different time without separating cause and effect and the causal relation can be established by the help of the time arrow as mentioned above. The solution of a local equation of motion is always causal because it can be obtained by direct integration where causality is guaranteed. But effective equations of motion are nonlocal and the memory term couples external sources to dynamical quantities of earlier in time, raising the possibility of causality. It was pointed out after Eq. (26) that the effective dynamics of our toy model with discrete spectrum remains causal but it is easy to see that the self energy (30) of continuous spectrum has poles with positive imaginary part for certain values of g and Ω_D.

The solution of the equation of motion of a harmonic system, eg. Eq. (24), can be written as $x=x_{ih}+x_h$, the sum of a particular solution of the inhomogeneous equation, x_{ih}, and a general solution of the homogeneous one, x_h. The latter belongs to the null-space of the linear equation of motion and is adjusted to satisfy the desired auxiliary conditions. Note that the null-space component of the trajectory drops out from the action thus this adjustment is carried out "by hand", by the choice of an infinitesimal imaginary part of the poles of the Green function, used to obtain x. This procedure is replaced in the CTP scheme by a functional definition of the retarded Green function, based on (19), where the imaginary part of the CTP action leads to a well defined pole structure.

The apparent conflict between acausality, arising in case of spectrum with condensation point and the trivially causal trajectory given by the direct integration can be resolved by recalling that any integration quadrature, based on the iteration $t\to t+\Delta t$ of a discretized version of Eqs. (21) implies a double limit, namely $N\to\infty$ and $\Delta t\to 0$. The continuum limit, $\Delta t\to 0$, carried out first leads to equations which are local in time and therefore causal. If the spectrum has condensation point then we have need continuous spectrum and we carry out the thermodynamical limit, $N\to\infty$, first. Let us suppose that we can solve the finite difference equations sequentially, for one coordinate after the other, for a given time. The solution of each second order finite difference equation extends the time interval from which the coordinates are used by $2\Delta t$, backward in time. Therefore such an equation of motion may become nonlocal and develop memory, allowing acausality.

F. Runaway solutions

It is not difficult to find spectral function to the self energy (36) which keeps the Hamiltonian bounded from below and produces the local equation of motion,

$$0=\gamma\dddot{x}-\ddot{x}+\omega_0^2x, \tag{38}$$

with solution,

$$x(t)=c_+\cos\omega_+t+c_-\cos\omega_-t+c_re^{\omega_rt}, \tag{39}$$

where $\omega_\pm=\pm\omega_1+i\omega_2$, $\omega_2=\mathcal{O}(\gamma)$, $\omega_r=+\mathcal{O}\left(\gamma^{-1}\right)$ are the roots of the equation $0=i\gamma\omega^3+\omega^2-\omega_0^2$ and $c_\pm$ and c_r are constant. How could a runaway solution and instability arise in a simple, stable harmonic model?

The runaway trajectory is indeed a true solution of Eq. (38) but the application of an effective equation of motion is more restricted than that of an elementary one as far as the auxiliary conditions are concerned. The toy model (23) has $N+1$ degree of freedom and the elimination of the environment uses $2N$ auxiliary conditions. We are thus

left with two auxiliary conditions to impose on the system coordinate and these are not enough to identify a solution the effective equation if there are higher order derivatives.

A safe solution to this problem can be found by imagining a thought experiment to observe the system. One starts at a sufficiently early time and let the system evolve under the influence of some external fields, prepared by the experimenter. These fields drive the system adiabatically to the desired state of motion by the beginning of the observation when they are turned off. This can easily be realized in a calculation by means of retarded Green functions, derived in the limit $t_f - t_i \to \infty$ as described in Section II C and the system initial conditions are now set by the external source.

The usual calculation of the retarded Green function of Eq. (38), based on the use of residuum theorem, shows that thought the Green function may not be causal nevertheless it remains bounded in time, $|D(t)| < \infty$, the runaway solution is used in the acausal contribution, for $t < 0$ only. In other words, the time arrow is introduced for a half time axis only and the time is allowed to run in that infinite half interval, either before or after observation, where the solution is bounded. The retarded Green function, obtained by means of the residuum theorem automatically excludes runaway solutions and the unstable modes signal acausality.

IV. POINT CHARGES

Let us now finally consider a system of point charges moving in the presence of an external electromagnetic field $A^e_\mu(x)$, described by the action

$$S[x,A] = -\sum_a \int ds \left(m_0 c\sqrt{\dot{x}_a^2(s)} + \frac{e}{c}\dot{x}_a^\mu(s)[A^e_\mu(x_a(s)) + A_\mu(x_a(s))]\right) + \frac{1}{16\pi c}\int dx F^{\mu\nu} Q(\Box) F_{\mu\nu} \tag{40}$$

where $x^\mu_a(s)$ denotes the world line of the a-th point charge, the parameter s is chosen to be the invariant length after the variation of the world line, $F_{\mu\nu} = \partial_\mu A_\mu - \partial_\nu A_\mu$ and an UV regulator Q is introduced, indicated a smearing of either the EM field or the current. An UV regulator has already been used electrodynamics [16] but the regulator, applied in this work satisfies more stringent conditions, specified below. The effective action for the charges [17],

$$S_{eff}[x] = -m_0 c \sum_a \int ds \left(\sqrt{\dot{x}_a^2(s)} + \frac{e}{c}\dot{x}_a^\mu(s) A^e_\mu(x_a(s))\right) - 2\pi\frac{e^2}{c}\sum_{a,b}\int ds ds' \dot{x}_a(s) D^n(x_a(s) - x_b(s'))\dot{x}_b(s'), \tag{41}$$

involves the symmetric near field Green function, $D^n(x) = D^n(-x) = -\delta_\ell(x^2)/4\pi$ which contains a regulated Dirac-delta and produces the equation of motion

$$m_0 c \ddot{x}_a = \frac{e}{c}(F^e + F^n)\dot{x}_a \tag{42}$$

where F^n and F^e denote the field strength tensor of the near field of the charge and the external field, respectively. The effective theory (41) does not contain radiation and a possible, non-conventional way to recover it is to assume that all electromagnetic radiation is finally absorbed [18]. Instead of following this line of thought we construct the effective theory within the framework of the CTP formalism where the issue of retardation is handled in an automatic manner.

A. CTP effective action

The full charge plus EMF system is described by the CTP action

$$S_{CTP}[\hat{x},\hat{A}] = S[x^+, A^+] - S^*[x^-, A^-], \tag{43}$$

where the single time axis action is taken over from Eq. (40), supplemented by the shift $S[x,A] \to S[x,A] - i\epsilon A^2/8\pi$. The time translation invariant action (15), used here for the EMF yields

$$\begin{aligned} S_{CTP}[\hat{x},\hat{A}] &= -\sum_{\sigma=\pm}\sigma\sum_a\int ds\left(m_0 c\sqrt{\dot{x}_a^{\sigma 2}(s)} + \frac{e}{c}\dot{x}_a^{\sigma\mu}(s)[A^{\sigma e}_\mu(x^\sigma_a(s)) + A^\sigma_\mu(x^\sigma_a(s))]\right) \\ &+ \frac{1}{8\pi c}\int\frac{dp}{(2\pi)^4}(A^+_\mu(-p), A^-_\mu(-p))\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)Q(-p^2)\begin{pmatrix} -p^2 - i\epsilon & 2i\epsilon\Theta(-p^0) \\ 2i\epsilon\Theta(k^0) & p^2 - i\epsilon\end{pmatrix}\begin{pmatrix} A^+_\nu(p) \\ A^-_\nu(p)\end{pmatrix}. \end{aligned} \tag{44}$$

The effective action

$$S_{eff}[\hat{x}] = -m_0 c \sum_a \int ds[\sqrt{\dot{x}_a^{+2}(s)} - \sqrt{\dot{x}_a^{-2}(s)}] + \frac{2\pi e^2}{c} \sum_{a,b} \int ds ds' \dot{\hat{x}}_a(s) \sigma \hat{D}(x_a(s) - x_b(s')) \sigma \dot{\hat{x}}_b(s'), \qquad (45)$$

can easily be obtained where the matrix σ is generated by the negative sign on the right hand side of Eq. (1) and the massless CTP Green function is given by the Fourier transform of Eq. (A25),

$$\mathrm{Re}\hat{D} = \frac{\delta_\ell(x^2)}{4\pi} \begin{pmatrix} -1 & \epsilon(x^0) \\ -\epsilon(x^0) & 1 \end{pmatrix}, \qquad (46)$$

where $\epsilon(x) = \mathrm{sign}(x)$. The effective action is now given by

$$\begin{aligned} S_{eff}[x] &= -m_0 c \sum_a \int ds \sqrt{\dot{x}_a^2(s)} + \frac{e^2}{2c} \sum_{a,b} \int ds ds' [\dot{x}_a^+ \delta_\ell((x_a^+ - x_b'^+)^2) \dot{x}_b'^+ - \dot{x}_a^- \delta_\ell((x_a^- - x_b'^-)^2) \dot{x}_b'^-] \\ &\quad + \frac{e^2}{c} \sum_{a,b} \int ds ds' \dot{x}_a^+ \epsilon(x_a^{+0} - x_b'^{-0}) \delta_\ell((x_a^+ - x_b'^-)^2) \dot{x}_b'^-, \end{aligned} \qquad (47)$$

where the notation $x = x(s)$, $x' = x(s')$ is employed.

The structure of the effective action (47) shows clearly the way interactions are organized in the CTP scheme. An elementary interaction between particle a and b contributes to the action by $\Delta S_{ab} = e^2 \dot{x}_a \dot{x}_b'/2$ up to a sign. The interactions which were called closed above are independent of the orientation of the time and are mediated by the symmetric, near field Green function. This is in agreement with the well known fact that the near field created by a point charge system remain localized and the energy-momentum of the charge plus its near field is preserved in the absence of outgoing radiation. The open interactions are described by the far field because the radiation field decouples from the charge. Their contributions to the influence functional in the second equation of (47) can be written as

$$\Delta S^f_{ab}(s, s') = \Delta S^f_a(s) - \Delta S^f_b(s'), \qquad (48)$$

where $\Delta S^f_c(s)$ corresponds to the action of the far field component of the Liénard-Wiechert potential, emitted by the charge c at the event, labeled by the value s of the invariant length . In fact, the effective action is just the action of the full charge and EMF system where the EMF is expressed in terms of the charge coordinates and the negative sign in (48) results from the relative minus sign of the contributions of the two time axes on the right hand side of Eq. (1). The two trajectories of the CTP doublet are matched at the end of the motion in the original, elementary description where both the charges and the EMF are present thus both time axes must have the same the EMF field at the final time. The contributions of the far field components of the two time axes cancel during the time when these field components are identical on the two time axes. Hence the far field contribution appears localized between its emission times on the two time axes and it appears as if the far field would leave the charge at a given moment and would be absorbed at another time, just as in the case of QED. It is remarkable that system-environment interactions which make the system open are odd with respect to time reversal and can transfer the environment time arrow to the system [4].

B. Regularization

Now we address the problem of UV singularities of the self-interaction of point charges. The regularization of the Dirac-delta can be done easily in the near field Green function but special care in needed in the case of the far field Green function where the singularities of the distributions $\delta(x^2)$ and $\epsilon(x^0)$ must be shifted away from each other. The problem can be seen by noting that contributions to the effective action, containing $\epsilon(x^0)$ are not Lorentz invariant and must be suppressed to recover a Lorentz invariant effective dynamics. This is a problem of point charges only because Lorentz invariance remains intact for continuous charge distribution. In fact, the limit $x^0 \to 0$ appears together with $\boldsymbol{x} \to 0$ in the effective action (47), because a pair of events with space-like separation plays no role in classical dynamics ($a \neq b$), and the interaction of continuously distributed charges mediated by the far field is regular, therefore it is negligible in sufficiently small volume ($a = b$).

The regulated Dirac-delta which is chosen to be equivalent in the near and far Green function to keep the EMF causal should satisfy the following conditions: The suppression of the divergence, arising from the time derivative acting on $\epsilon(x^0)$ requires

$$\delta_\ell(0) = 0. \qquad (49)$$

Due to Lorentz invariance the Dirac-delta of the far field Green function assumes the same value at space-time events visited by the massless radiation field. This value must obviously be non-vanishing which together with the property (49) forces us to shift the support of the regulated Dirac-delta slightly off light-cone. To recover the correct energy-momentum flux of the radiation field we need

$$\int_0^a dx\delta_\ell(z) = 1 \tag{50}$$

for any fixed $a > 0$ as $\ell \to 0$. Lorentz invariance requires the separation of the future and the past components of the support of the regulated Green function in such a manner that each of them have a unit weight at any space separation. A possible regulated Dirac-delta, satisfying these conditions is of the form

$$\delta_\ell(z) = \frac{\Theta(z)}{12\ell^4} z e^{-\frac{\sqrt{z}}{\ell}}. \tag{51}$$

Note that the condition (50) solves another problem of the self-force: One understands the factor 2π in front of the second term in the right hand side of Eq. (41) as $1/2$ times 4π where the factor half represents the double counting of interactions by the independent summation over a and b. But this generates an unwanted factor half for self interaction. This factor is compensated by integrating over both the past and the future light cone in the calculation of the Liénard-Wiechert potential of a point charge.

C. Lagrangian of a single charge

We are interested in the radiation reaction force therefore it is sufficient to consider the effective action (47) for a single charge. Thought the external field F^e is an essential ingredient to induce radiation reaction in case of a single charge it appears in the equation of motion in a trivial manner and will be suppressed. It is advantageous to use the parametrization $x^\pm = x \pm x^d/2$ because we know that $x^d = 0$ holds for the solution thus it is sufficient to calculate the effective action up to $\mathcal{O}\left(x^d\right)$. The free action is of the form

$$S_{free}[x] = -\sum_{\sigma=\pm} \sigma m_0 c \int ds \sqrt{\left(\dot{x} + \frac{\sigma}{2}\dot{x}^d\right)^2} \tag{52}$$

and the near and far field contributions to the influence functional are

$$\begin{aligned} S^n_{eff} &= -\frac{e^2}{c}\int dsds'[\dot{x}^d\dot{x}'\delta_\ell(R^2) + \dot{x}\dot{x}'\delta'_\ell(R^2)RR^d], \\ S^f_{eff} &= -\frac{e^2}{c}\int dsds'\{2\dot{x}\dot{x}'[\epsilon(R^0)Rx^d\delta'_\ell((R^2)) + x^{d0}\delta(R^0)\delta_\ell((R^2)] - \dot{x}^d\dot{x}'\epsilon(R^0)\delta_\ell((R^2)\}, \end{aligned} \tag{53}$$

respectively where $R = x' - x$ and $R^d = x'^d - x^d$. The dominant contribution to the integrals comes from $s \sim s'$ and the non-locality of the dynamics is $\mathcal{O}(\ell)$. We write $\dot{x}' \approx \dot{x} + u\ddot{x} + u^2\dddot{x}/2$, $R \approx u\dot{x} + u^2\ddot{x}/2 + u^3\dddot{x}/6$ and $R^2 \approx u^2 - u^4\ddot{x}^2/12$ where $u = s' - s$ and introduce the moments

$$a_{j,k} = \int_{-\infty}^{\infty} du\delta^{(j)}_\ell(u^2)|u|^k = c_{j,k}\ell^{k+1-2(j+1)}, \tag{54}$$

where $\delta^{(j)}_\ell(z) = d^j\delta_\ell(z)/dz^j$. The relation $a_{j+1,k+2} = -(k+1)a_{j,k}/2$ holds for arbitrary value of ℓ and the property $a_{j,2k+1} = \delta_{j,k}(-1)^j j!$ is recovered in the limit $\ell \to 0$.

The effective Lagrangian will be calculated in two approximations, first by ignoring $\mathcal{O}(\ell)$ terms which vanish when the cutoff is removed but keeping high powers of x, next by skipping $\mathcal{O}\left(x^3\right)$ anharmonic contributions but keeping all cutoff dependence in the quadratic Lagrangian. A straightforward calculation gives the $\mathcal{O}\left(x^d\right)$ Lagrangian

$$L_{eff} = x^d\left[m_0 c\ddot{x} + \frac{c_{0,0}e^2}{2c^2\ell}\ddot{x} - \frac{2e^2}{3c}(\dddot{x} + \ddot{x}^2\dot{x})\right] \tag{55}$$

up to vanishing contribution as $\ell \to 0$ after ignoring total s-derivatives. The omitted terms which vanish in the limit $\ell \to 0$ include higher order derivatives and non-linear combinations of x.

To find the full cutoff-dependence of the quadratic action we return to the influence functional (53) and write the quadratic part of the corresponding Lagrangian as

$$L_{eff} = x^d \left\{ m_0 c\ddot{x} + \frac{4e^2}{c}\int_{-\infty}^{0} du \delta'_\ell(u^2)[x(s+u) - u\dot{x}(s+u) - x] \right\} \tag{56}$$

after ignoring total derivatives and terms $\mathcal{O}\left(x^{d2}\right)$ and $\mathcal{O}\left(x^3\right)$, where $\delta'(a) = d\delta(z)/dz$ and the argument s is suppressed, $x(s) = x$. The cancellation between the near and far components of the self energy is complete in the future for $\ell \neq 0$, leaving behind a memory term and a Volterra-type integro-differential equation of motion. A simple power counting is sufficient to find the behavior in the limit $\ell \to 0$, the dominant contribution to the integral comes from $u = \mathcal{O}(\ell)$ and the order of magnitude $\delta'_\ell(u^2) = \mathcal{O}\left(\ell^{-4}\right)$ shows that $\mathcal{O}(u^n)$ terms are divergent when $n \leq 2$, finite for $n = 3$ and vanishing for $n \geq 4$ as $\ell \to 0$. These contributions are separated by writing Eq. (56) as

$$L_{eff} = x^d \left\{ m_0 c\ddot{x} + \frac{c_{0,0}e^2}{2c^2\ell}\ddot{x} - \frac{2e^2}{3c}\dddot{x} + \frac{4e^2}{c}x^d \int_{-\infty}^{0} du \delta'_\ell(u^2)\left[x(s+u) - u\dot{x}(s+u) - x + \frac{u^2}{2}\ddot{x} + \frac{u^3}{3}\dddot{x}\right]\right\}, \tag{57}$$

where the square bracket in the integrand approaches zero sufficiently fast at $u = 0$ to make the whole integral vanishing as $\ell \to 0$.

D. Renormalization

A divergent, $\mathcal{O}\left(\ell^{-1}\right)$ part of the Lagrangians obtained above can be treated as a mass renormalization in a manner similar to quantum field theories. The the renormalization, the removal of the cutoff is carried out by keeping the $\mathcal{O}(\ddot{x})$ term of the Lagrangian cutoff-independent, ie. by making the bare mass cutoff-dependent,

$$m_0(\ell) = m - \frac{c_{0,0}e^2}{2c^2\ell}, \tag{58}$$

where m is the observed, physical mass. The resulting equation of motion,

$$mc\ddot{x} = \frac{e}{c}F^e\dot{x} + TK \tag{59}$$

where the external field F^e is reintroduced again, involves

$$K = \frac{2e^2}{3c}\dddot{x} - \frac{4e^2}{c}\int_{-\infty}^{0} du \delta'_\ell(u^2)\left[x(s+u) - u\dot{x}(s+u) - x + \frac{u^2}{2}\ddot{x} + \frac{u^3}{3}\dddot{x}\right] + \mathcal{O}\left(x^2\ell\right), \tag{60}$$

where the projector $T^{\mu\nu} = g^{\mu\nu} - \dot{x}^\mu\dot{x}^\nu$, arising from the expansion of the regulated Dirac-delta, projects on the transverse subspace of $\dot{x}$. The the cutoff-independent, $\mathcal{O}\left(\ell^0\right)$ Abraham-Lorentz force comes from the far field component of the self energy, the first term in the right hand side of the second equation of (53) and its sign changes if the time arrow of the EMF is flipped, indicating that this force is generated by the spontaneous breakdown of partial time reversal invariance and represents the time arrow transferred dynamically from the EMF to the charge. The equation of motion (59) is linear except the projector, which represents a modest partial resummation of the $\mathcal{O}\left(x^2\ell\right)$ terms, arising from the expansion of the regulated Dirac-delta.

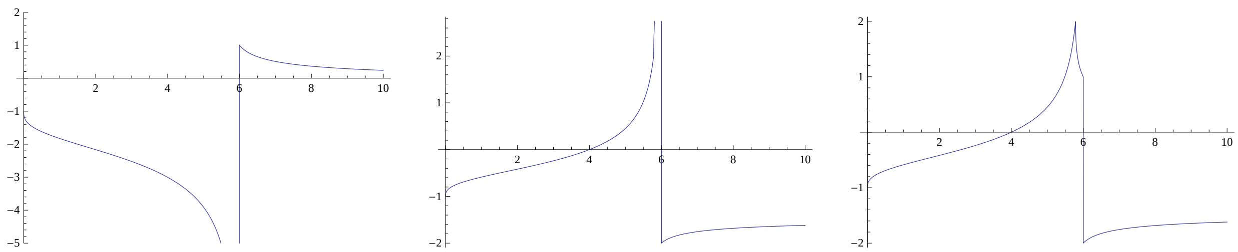
Note that the Lagrangian (56) does not contain higher order derivative term for $\ell \neq 0$. The first term on the right hand side of Eq. (60) survives the limit $\ell \to 0$ due to the singularity of the memory kernel, the factor $\delta'(u^2)$ of the integrand. The cutoff-independent Abraham-Lorentz force is generated by the nonuniform convergence of the memory term of the effective Lagrangian in the limit $\ell \to 0$. The convergence must be non-uniform because the EMF Green function is a generalized function, distribution, which can not lead to uniformly convergent integral.

To find physically better motivated parameters we evaluate the Lagrangian (56) for the world lines $x(s) = x_0 e^{-i\omega s}$ and $x^d(s) = x^d$. The choice of the regulated Dirac-delta (51) which gives $c_{0,0} = 1/3$ leads to

$$L_{eff} = -x^d x_0 m_0 c\omega^2 \left[1 + \frac{\lambda_0}{6}\frac{1 + i\ell\omega}{(1 - i\ell\omega)^3}\right] \tag{61}$$

with $\lambda_0 = e^2/m_0c^2\ell$. One can write this expression by absorbing the cutoff-dependence into the renormalized parameters as

$$L_{eff} = -x^d x_0 mc\omega^2 \left[1 + \lambda\left(\frac{2}{3}r_0 i\omega + r_0\omega\mathcal{O}(\ell\omega)\right)\right] \tag{62}$$



(a) (b) (c)

FIG. 1: The imaginary part of the poles, (a) $\ell\mathrm{Im}\omega_1$, (b) $\ell\mathrm{Im}\omega_+$ and (c) $\ell\mathrm{Im}\omega_-$ as functions of $\lambda = r_0/\ell$.

where $m = m_0(1+\lambda_0/6)$ and

$$\lambda = \frac{\lambda_0}{1+\frac{\lambda_0}{6}} = \frac{e^2}{mc^2\ell}, \tag{63}$$

denotes the classical charge radius expressed in units of the UV cutoff. It plays the role of a renormalized strength of self interaction, described by the Abraham-Lorentz force in the renormalized theory. The bare coupling constant,

$$\lambda_0 = \frac{\lambda}{1-\frac{\lambda}{6}}, \tag{64}$$

as the function of the cutoff possesses a pole in as in case of the Landau-pole of perturbative QED where the diverging coupling constant is induced at the Landau pole by the one-loop photon self energy. But there is no EMF self-energy for fixed charge world lines and our singularity arises due to the mass renormalization.

The parametrization (34) of the action allows us to read off the retarded Green function of the Lagrangian (56), describing the response of a non-relativistic charge with quadratic free Lagrangian on space independent external field,

$$D^r(t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{e^{-i\omega t}}{\omega^2+\frac{\lambda_0(\omega^2+i\ell\omega^3)}{6(1-i\omega\ell)^3}}. \tag{65}$$

The integrand has five poles, two of them are in the vicinity of zero and give a term proportional to the time as $\ell \to 0$, describing the free, causal motion of the charge. The remaining roots are

$$\begin{aligned} \omega_1 &= -\frac{i}{\ell}\left(1+\frac{\lambda_0}{3\sqrt[3]{2}q}+\frac{q}{3\sqrt[3]{4}}\right), \\ \omega_\pm &= -\frac{i}{\ell}\left(1-\frac{\lambda_0(1\pm i\sqrt{3})}{6\sqrt[3]{2}q}-\frac{(1\mp i\sqrt{3})q}{6\sqrt[3]{4}}\right). \end{aligned} \tag{66}$$

with $q = \sqrt[3]{\lambda_0(18+\sqrt{324-2\lambda_0})}$. The imaginary parts of the poles are shown in Fig. 1. The charge dynamics is causal for $\ell > r_0/4$ but the exact position of the loss of causality naturally depends on the choice of the regulator. A pole is on the "wrong" sheet for $\ell < r_0/4$ but the runaway modes do not lead to unstable motion according to Section III F, the Green function turns the runaway modes off at the time of observation. Both the real and the imaginary parts of the poles display a singularity at the classical Landau-pole, $\ell = r_0/6$.

It is natural that a regulated, cutoff theory shows unusual, nonphysical features at the cutoff scale, represented by the modes with finite $\ell\omega$ in our case, eg. $\omega_\pm = (\pm\sqrt{7}-3i)/2\ell$ as $\ell \to 0$. The reflection of the cutoff-independence of the Abraham-Lorentz force, mentioned above is that the frequency of the mode, representing this force has a cutoff-independent, finite limit, $\omega_1 \to 3i/2r_0$ in the renormalized theory. The radiation reaction force extends acausality beyond the UV cutoff length, to the internal length scale, r_0.

E. Classical anomaly

The effective dynamics of point charges, outlined above, displays remarkable UV sensitivity which arises from two sources. First, the EMF is divergent at the location of a point charge and second, certain manifestations of the

radiation reaction force can be viewed as a magnifying process. In fact, energy-momentum conservation correlates the short distance structure of the EMF around the charge with the radiation field. The latter decouples from its source and carries this information over large distances. Such a sensitivity of the radiation field on the microscopic details of the charge distribution explains the inconsistencies, found in direct calculation of the Lorentz force [19] and its inference by means of energy-momentum conservation [2]: The reaction force acquires an artificial multiplicative factor $4/3$ in the limit $\ell \to 0$ when the narrow world-tube of the almost point-like charge lacks Lorentz contraction.

The magnification effect of the radiation field stems from the absence of ingoing spherical waves in the radiation field, from the radiation time arrow. This time arrow generates a friction force in the effective charge dynamics, reflecting the dissipation of the energy of the charge to infinitely many IR EMF modes and explains the IR sensitivity of the massless EMF dynamics. In fact, the restriction of the EMF into an arbitrary large but finite volume renders its spectrum discrete and restores reversible effective dynamics.

UV singularities are known to play role in quantum effects of finite scales and this mechanism has a somehow unfortunate name, anomaly, because it was not expected that phenomena with very different scales can be coupled to each others. The cutoff is sent beyond any observation scale in the renormalization procedure and drops out from relations among observed quantities. It was therefore surprising that the presence of an arbitrarily distant cutoff leaves a finite trace in relations among observables. For instance, the rule of partial integration in the action is modified for a non-relativistic particles [20]. Such an $\mathcal{O}(\hbar)$ effect of the canonical commutation relation, a quantum anomaly, is due to the nowhere differentiable fractal nature of trajectories which dominate the path integral expressions of non-relativistic quantum mechanics.

The diverging number of one particle states leads further UV singularities in quantum field theories where certain symmetries, such as scale invariance [21] or chiral symmetry [22] are not preserved. Such anomalies result in perturbative quantum field theory from the non-uniform convergence of loop-integrals as the UV cutoff is removed. The order of the integration and the removal of the cutoff can be exchanged without modifying the integral as long as the loop-integral is uniformly convergent. One can first perform the subtraction of the BPHZ renormalization scheme on the integrand itself [23] in this case, leaving behind no cutoff effects in the renormalized perturbation expansion. But loop-integrals without uniform convergence which are accidentally UV finite and receive no regulating counterterm may converge in a non-uniform manner and retain finite cutoff effects even for arbitrarily large values of the cutoff. Quantum anomalies arise in similar fashion, as well [24].

The sensitivity of the reaction force on the details of the charge distribution in the point-like limit, such as the factor $4/3$, mentioned above is a classical UV anomaly in this sense. The Abraham-Lorentz force is anomalous, too, because it results from the non-uniform convergence of the only loop-integral of the effective theory, according to the remark, made after Eq. (60).

Lorentz symmetry might anomalously be broken in classical electrodynamics of point charges and our regulator had to be carefully chosen to avoid the loss of relativistic symmetries. This is reminiscent of the Schwinger term in quantum field theory, the anomalous contribution to the commutator of global Noether currents which arises from the time ordering in the Feynman propagator [25]. Anomalous Schwinger term appears in classical field theory, as well, when the commutator is replaced by the Poisson bracket [26]. The divergence encountered in this work can be considered as a generalization of such singularities beyond conserved currents.

V. CONCLUSIONS

The effective action was constructed in this work for point charges in classical electrodynamics. To accommodate the dynamical breakdown of time reversal invariance and the resulting dissipative forces in the action principle the CTP formalism was used in classical setting. This scheme (i) distinguishes the non-time reversal invariant contributions coming from the initial conditions and from the equations of motion, (ii) offers a simple view of the origin of irreversibility in effective theories and (iii) separates clearly the near and far field components of EMF. This latter feature leads to two separate classes of the interactions in the effective theory of charges, those which remain localized to the charge system from those which decouple from the charges. Since CTP does not impose constraint on the time evolution and can handle initial condition problems this version of the action principle can easily handle the radiation field, the key ingredient of the reaction force.

The well known Abraham-Lorentz force and the divergent renormalization of the mass have been reproduced and the linearized equation of motion was obtained by retaining the full cutoff dependence. The similar origin of the Abraham-Lorentz force and anomalies has been pointed out, namely these are cutoff-independent effects due to the non-uniform convergence of loop-integrals. The strength of the radiation reaction force displays a pole in its cutoff dependence in a manner reminiscent of the Landau pole.

It is argued that as soon as higher order derivatives appear in an effective equation of motion then the solution can not be found by imposing auxiliary conditions, rather one has to use Green functions. When these latter are obtained

by the application of the residuum theorem then they always describe stable motion, without runaway solutions. The runaway mode, appearing at $\ell \ll r_0$ lead to acausality at the scale r_0.

Quantum effects pose further challenges. The classical argument is based on a double limit, both the Planck constant and the size of the charge distribution or the UV cutoff, ℓ, are sent to zero. These limits do not commute and the wrong order is followed whenever the point charge limit is pursued in classical electrodynamics. The equation of motion, derived above remains the same for the expectation value of the charge coordinate, as far as non-relativistic motion is concerned. In fact, the classical linear equations, derived from a quadratic action are satisfied in the quantum case by expectation values. Therefore true quantum corrections without classical counterpart show up through world lines with non-monotonous dependence of time in the proper time which correspond particle-anti particle pair creation. It remains to be seen how do such self-energy corrections correct the radiation reaction force in the effective path integral description of the charge.

Appendix A: CTP Green function

The 2×2 CTP block Green function is derived in this Appendix first for a harmonic oscillator [4], a massive scalar field and the EMF.

1. Harmonic oscillator

The simplest building block in constructing the Green function for free fields is a single harmonic oscillator, described by the action

$$S[x] = \int dt \left(\frac{m}{2}\dot{x}^2 - \frac{m\Omega^2}{2}x^2\right). \tag{A1}$$

We follow its time evolution in discrete steps, by taking $t_j = j\Delta t - T$, $j = 0, \ldots, N$, with $\Delta t = T/N$. The time interval of the motion is chosen to be $[-T, 0]$ in order to remove ambiguities at the final final in the limit $T \to \infty$. The CTP action (1),

$$S_{CTP}[x^+, x^-] = \frac{1}{2}\sum_{\sigma,\sigma'}\sum_{\Delta t - T \le t,t' < 0} x^\sigma_t (\hat{D}_0^{-1})^{\sigma,\sigma'}_{t,t'} x^{\sigma'}_{t'} + \sum_{\sigma}\sum_{\Delta t - T \le t < 0} x^\sigma_t \hat{A}^\sigma_t z \tag{A2}$$

where $z = x_0$ and

$$\begin{aligned}(\hat{D}_0^{-1})^{\sigma,\sigma'}_{t,t'} &= -\delta^{\sigma,\sigma'}\left[\sigma\left(\frac{m}{\Delta t}(\delta_{t,t'+\Delta t} + \delta_{t,t'-\Delta t} - 2\delta_{t,t'}) + \Delta t m\Omega^2\delta_{t,t'}\right) - i\Delta t m\epsilon\delta_{t,t'}\right],\\ \hat{A}^\sigma_t &= -\delta_{t,-\Delta t}\frac{\sigma m}{2\Delta t}.\end{aligned} \tag{A3}$$

can be written as

$$S[\hat{x}] = \frac{1}{2}\hat{x}\hat{D}_0^{-1}\hat{x} + \hat{x}\hat{A}z. \tag{A4}$$

The calculation of the Green function, the quadratic form of the effective action for $\hat{x}$ can be carried out by finding the functional $W[\hat{j}] = S[\hat{x}] + \hat{x}\hat{j}$, cf. Eq. (16). First we eliminate $\hat{x}$ by means of its equation of motion, $\hat{x} = -\hat{D}_0(\hat{A}z + \hat{j})$, yielding

$$W[\hat{j}] = -\frac{1}{2}z\hat{A}\hat{D}_0\hat{A}z - z\hat{A}\hat{D}_0\hat{j} - \frac{1}{2}\hat{j}\hat{D}_0\hat{j}. \tag{A5}$$

Next we eliminate z by means of its equation of motion, $z = -\hat{A}\hat{D}_0\hat{j}/\hat{A}\hat{D}_0\hat{A}$, to find

$$W[\hat{j}] = -\frac{1}{2}\hat{j}\hat{D}\hat{j} \tag{A6}$$

with

$$\hat{D} = \hat{D}_0 - \hat{D}_0\hat{A}\frac{1}{\hat{A}\hat{D}_0\hat{A}}\hat{A}\hat{D}_0. \tag{A7}$$

The detailed expression for this Green function comes from the representation

$$x(t)=\sqrt{\frac{2}{T}}\sum_{n=0}^{N}\tilde{x}_n\sin\omega_n t \tag{A8}$$

of discrete trajectories $x(-T)=x(0)=0$ defined at $t=j\Delta t$, $j=-N,\ldots,-1$ where $\omega_n=\frac{\pi}{T}n$. For the construction of the Green function (A7) we need

$$D_0(t,t')=\frac{2}{Tm}\sum_{n=1}^{N}\frac{\sin\frac{\pi}{T}nt\sin\frac{\pi}{T}nt'}{\frac{4}{\Delta t^2}\sin^2\pi\frac{\Delta tn}{2T}-\Omega^2+i\epsilon}, \tag{A9}$$

and its special values,

$$D_0(t,-\Delta t)=-\frac{2}{Tm}\sum_{n=1}^{N}\frac{\sin\frac{\pi}{T}nt\sin\frac{\pi}{T}n\Delta t}{(\frac{\pi}{T}n)^2-\Omega^2+i\epsilon}, \tag{A10}$$

and

$$D_0(-\Delta t,-\Delta t)=\frac{2}{Tm}\sum_{n=1}^{N}\frac{\sin^2\frac{\pi}{T}n\Delta t}{\frac{4}{\Delta t^2}\sin^2\pi\frac{\Delta tn}{2T}-\Omega^2+i\epsilon}. \tag{A11}$$

We remove first the UV cutoff by carrying out the limit $N\to\infty$ which is not uniform owing to the non-differentiability of the CTP trajectory at the final time. This is followed by the removal of the IR cutoff, $T\to\infty$, yielding

$$\begin{aligned}D_0(t,t') &= \frac{1}{2\pi m}\int_{-\infty}^{\infty}d\omega\frac{e^{i\omega(t-t')}-e^{i\omega(t+t')}}{\omega^2-\Omega^2+i\epsilon}\\ &= -\frac{i}{2m\Omega}\left[e^{-(i\Omega+\frac{\epsilon}{2\Omega})|t-t'|}-e^{(i\Omega+\frac{\epsilon}{2\Omega})(t+t')}\right],\end{aligned} \tag{A12}$$

for $t,t'<0$, together with

$$\begin{aligned}D_0(t,-\Delta t) &= \frac{i\Delta t}{\pi m}\int_{-\infty}^{\infty}d\omega\frac{\omega e^{i\omega t}}{\omega^2-\Omega^2+i\epsilon}\\ &= -\frac{\Delta t}{m}e^{(i\Omega+\frac{\epsilon}{2\Omega})t}\end{aligned} \tag{A13}$$

and

$$\begin{aligned}D_0(-\Delta t,-\Delta t) &= \frac{2\Delta t}{m\pi}\int_0^{\pi}d\omega\left(1-\sin^2\frac{\omega}{2}\right)+\frac{2\Delta t^2\Omega^2}{m\pi}\int_0^{\infty}d\omega\frac{1}{\omega^2-\Omega^2+i\epsilon}\\ &= \frac{\Delta t}{m}-i\frac{\Delta t^2\Omega}{m}.\end{aligned} \tag{A14}$$

The initial condition decouples after sufficiently long time evolution if some relaxation is present but the final condition on the CTP trajectories remains always relevant, motivating the use of the time interval $-T<t<0$. The full CTP propagator, given by (A7),

$$\hat{D}(t,t')=-\frac{i}{2m\Omega}\begin{pmatrix}e^{-i\Omega|t-t'|} & e^{i\Omega(t-t')}\\ e^{-i\Omega(t-t')} & e^{i\Omega|t-t'|}\end{pmatrix}, \tag{A15}$$

recovers translation invariance in time. Its Fourier transform,

$$\hat{D}(\omega)=\int dte^{i\omega t}\hat{D}(t,0), \tag{A16}$$

turns out to be

$$\hat{D}(\omega)=\frac{1}{m}\begin{pmatrix}\frac{1}{\omega^2-\Omega^2+i\epsilon} & -2\pi i\Theta(-\omega)\delta(\omega^2-\Omega^2)\\ -2\pi i\Theta(\omega)\delta(\omega^2-\Omega^2) & -\frac{1}{\omega^2-\Omega^2-i\epsilon}\end{pmatrix}. \tag{A17}$$

The inverse Green function can be obtained by means of the representation $\delta_\epsilon(\omega)=\omega/\pi(\omega^2+\epsilon^2)$ of the Dirac-delta in Eqs. (12)-(13),

$$\hat{D}_0^{-1}(\omega)=m\sigma\left[(\omega^2-\Omega^2)\begin{pmatrix}1&0\\0&-1\end{pmatrix}+i\epsilon\begin{pmatrix}1&2\Theta(-\omega)\\2\Theta(\omega)&1\end{pmatrix}\right]\sigma, \tag{A18}$$

or

$$D_0^{-1n}=\omega^2-\Omega^2,\quad D_0^{-1f}=i\mathrm{sign}(\omega)\epsilon,\quad D_0^{-1i}=\epsilon. \tag{A19}$$

2. Scalar field

The Green function can easily be found for free field in a similar manner. The expression (A7) for the Green function remains valid with

$$\hat{A}_t^\sigma(\boldsymbol{p})=-\delta_{t,-\Delta t}\frac{\sigma}{2\Delta t} \tag{A20}$$

given for the Fourier mode $\boldsymbol{p}$ and

$$D_0((t,\boldsymbol{x}),(t',\boldsymbol{x}'))=\frac{2}{Tm}\int\frac{d^3p}{(2\pi)^3}e^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{x}')}\sum_{n=1}^{N}\frac{\sin\frac{\pi}{T}nt\sin\frac{\pi}{T}nt'}{\frac{4}{\Delta t^2}\sin^2\pi\frac{\Delta tn}{2T}-\Omega^2(\boldsymbol{p})+i\epsilon}, \tag{A21}$$

where $\Omega(\boldsymbol{p})=\sqrt{m^2+\boldsymbol{p}^2}$ for a scalar field of mass m, in units $c=\hbar=1$. The limit $T\to\infty$ gives for the Fourier transform

$$\hat{D}(p)=\int dte^{ipx}\hat{D}(x) \tag{A22}$$

the result

$$\hat{D}(p)=\begin{pmatrix}\frac{1}{p^2-m^2+i\epsilon}&-2\pi i\Theta(-p^0)\delta(p^2-m^2)\\-2\pi i\Theta(p^0)\delta(p^2-m^2)&-\frac{1}{p^2-m^2-i\epsilon}\end{pmatrix}. \tag{A23}$$

The inverse Green function can be written as

$$\hat{D}^{-1}=(p^2-m^2)\begin{pmatrix}1&0\\0&-1\end{pmatrix}+i\epsilon\begin{pmatrix}1&-2\Theta(-p^0)\\-2\Theta(p^0)&1\end{pmatrix}. \tag{A24}$$

3. Electromagnetic field

The transverse part of the EMF Green function is given by a scalar massless Green function up to a sign,

$$\hat{D}^{\mu\nu}(p)=-\left(g^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right)\begin{pmatrix}\frac{1}{p^2+i\epsilon}&-2\pi i\Theta(-p^0)\delta(p^2)\\-2\pi i\Theta(p^0)\delta(p^2)&-\frac{1}{p^2-i\epsilon}\end{pmatrix}. \tag{A25}$$

The longitudinal part depends on the gauge fixing and drops out from the effective action due to current conservation.

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